$$\therefore\ F_{K_+} = (-A)^{-3\omega_0-3}\langle K_+\rangle \Rightarrow (-A)^{-3\omega_0}\langle K_+\rangle = (-A)^{3} F_{K_+}$$

$$F_{K_-} = (-A)^{-3\omega_0+3}\langle K_-\rangle \Rightarrow (-A)^{-3\omega_0}\langle K_-\rangle = (-A)^{-3} F_{K_-}$$

$$F_{K_0} = (-A)^{-3\omega_0}\langle K_0\rangle$$

and we have

$$A^4 F_{K_+} - A^{-4} F_{K_-} - (A^{-2} - A^{2}) F_{K_0} = 0$$

note: if we set $V_K(t) = F_L(t^{-1/4})$ then we see V_K satisfies

A) V_K an invariant of isotopy class of K

B) $t^{-1} V_{K_+} - t\, V_{K_-} - (t^{1/2} - t^{-1/2}) V_{K_0} = 0$

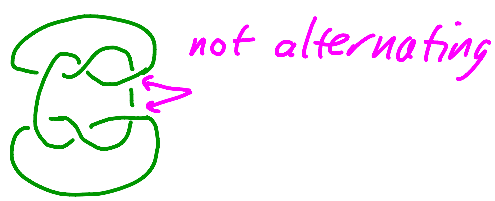
C) $V_{unknot} = 1$

i.e. V_K is the Jones polynomial!

and now we know it is well-defined!

E. Alternating Links

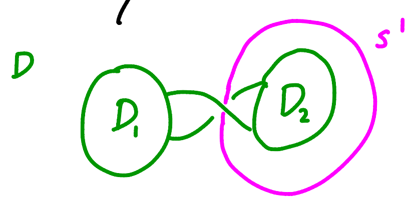
a knot diagram D is called alternating if over and under crossings alternate as you traverse the knot



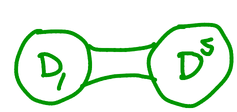
a link is alternating if it has an alternating diagram

an alternating diagram is called reduced if there is no embedded circle in $\mathbb{R}^2$ intersecting the diagrame transversely one time at a crossing

not reduced!



can remove this crossing $\Rightarrow$



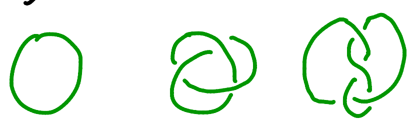
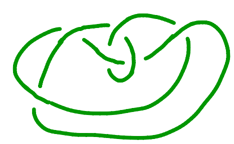
<u>exercise</u>: Show if D is an alternating diagram and it is not reduced then a sequence of "flips" as above will give a diagram that is reduced and alternating (with fewer crossings)

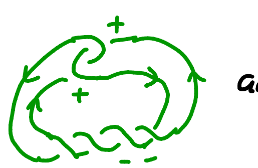
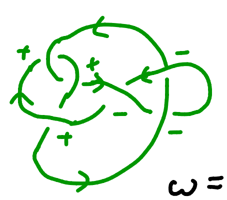
In ~ 1890 Tait conjectured the following two results (and a 3^{rd})

<u>Th$^{\underline{m}}$5</u>:

> I) If L has a reduced alternating diagram D then for <u>any</u> other diagram D' for L
>
> $$\#\text{ crossings of } D' \geq \#\text{ crossings of } D$$
>
> in particular L knotted! (unless D has no crossings)
>
> II) If D and D' are reduced alternating diagrams for an oriented link L then $\quad \omega(D) = \omega(D')$

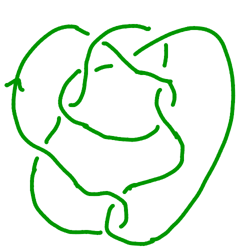
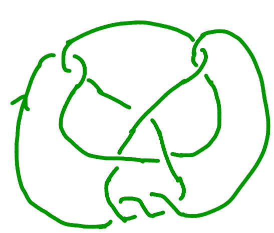
These are Amazing! with no more work we know


and

are different by I)

and

$\omega = -2$
and

$\omega = 0$
are different by II (even upto mirroring!)

<u>Remark</u>:

i) II not true in general

$\omega = 10$


$\omega = 8$

each has 10 crossings ("10_{161}" and "10_{162}")
but knots isotopic! (try to show this)

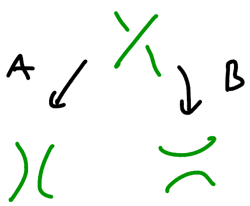
these knots we thought for decatedes to be different then Perko in 1970 showed they were the same! called the Perko Pair

2) above theorem was proven in 1980's after the discovery of the Jones polynomial

Proven by Kauffman, Murasugi, and Thistlethwaite (independently!)

We will prove I, but II is somewhat similar (though need better polynomial, see lecture suppliment)

Recall for a state s of a diagram D



$\alpha(s)$ = number of A-smoothings of s

$\beta(s)$ = number of B-smoothings of s

$|s|$ = number of components of s

and $$\langle D \rangle = \sum_{\substack{\text{all states} \\ s \text{ of } D}} A^{\alpha(s)-\beta(s)}(-A^{-2}-A^{2})^{|s|-1}$$

let s_A be the state with all A-smoothings and s_B similar for B

call a diagram D, A-adequate if
$$|s_A| > |s|$$
for all states s with $\alpha(s) = \alpha(s_A) - 1$ i.e. switch one crossing to a B

similarly define B-adequate

call D adequate if it is both A and B adequate

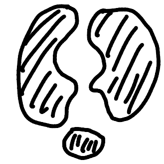
lemma 6:

$\boxed{\text{A reduced alternating diagram is adequate}}$

Proof: Given a diagram you can consider the "checker board" colloring of the plane

e.g.

so we have the black surface

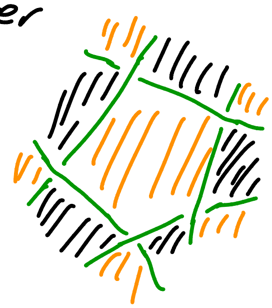


and the orange

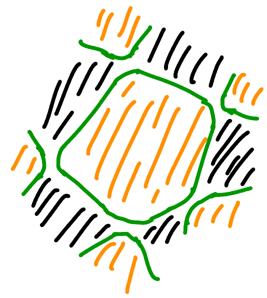
exercise: If D is alternating then s_A is the boundary of one of these surfaces and s_B the other

Hint: Consider

part of a diagram

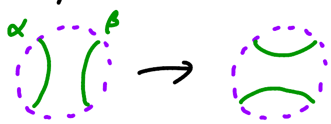


$\xrightarrow{\text{all } B \text{ splitting}}$



now let s be any state with one less B-splitting than s_B

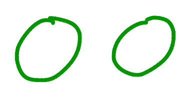
so the only difference is at one place where



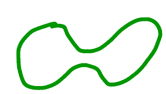
s_B $\qquad$ s

if the strands α and β are on different circles

then $|s| < |s_B|$



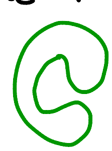
in s_B



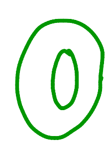
in s

if α and β are on same circle then

$|s| > |s_B|$



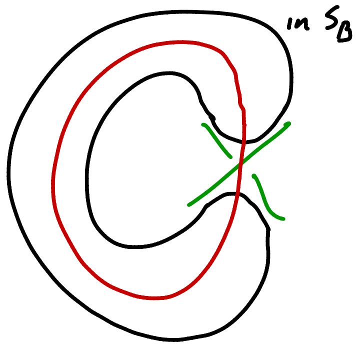
in s_B



in s

so we need to see second case can't happen

recall circles in s_B bound disks (orange or black) disjoint from D so if case 2 happens we see



so we can find an embedded S^1 showing D not reduced $\not\!\!\!\bigcirc$

given a Laurent polynomial $f(t)$ denote

$\max f$ = maximal degree of t in f
$\min f$ = minimal " " " "

lemma 7:

> let D be a connected diagram with n crossings
>
> I) $\max\langle D\rangle \leq n + 2|S_A| - 2$ with = if D is A adequate
>
> $\quad\min\langle D\rangle \geq -n - 2|S_B| + 2$ " " B adequate
>
> II) $|S_A| + |S_B| \leq n+2$ " " is alternating

we prove this later, but now

Th^{m} 8:

> let D be a connected, n-crossing diagram of a link L and $V_L(t)$ its Jones polynomial
>
> Then $\max V_L - \min V_L \leq n$ with equality if D is alternating and reduced.

called breadth of V_L denoted $Br V_L$

Clearly Tait I follows!

Proof: recall substituting $t = A^{-4}$ into $(-1)^{-3w(D)}\langle D\rangle$ gives $V_L(t)$

so
$$
\begin{aligned}
4 Br V_L = Br\langle D\rangle &= \max\langle D\rangle - \min\langle D\rangle \\
&\leq n + 2|S_A| - 2 - (-n - 2|S_B| + 2) \\
&= 2n + 2(|S_A| + |S_B|) - 4 \\
&\leq 2n + 2(n+2) - 4 = 4n \quad ✓
\end{aligned}
$$

if D is alternating and reduced, then it is adequate
so lemma 7.I) says $1^{st} \leq$ is $=$
lemma 7.II) says $2^{nd} \leq$ is $=$ so done!

Proof of lemma 7:

I) for a state s set $\langle s\rangle = A^{\alpha(s)-\beta(s)}(-A^2-A^{-2})^{|s|-1}$

so $\langle D\rangle = \sum_s \langle s\rangle$

largest term $A^{\alpha-\beta}(-A^2)^{|s|-1}$

now $\alpha(s_A) = n$ and $\beta(s_A) = 0$

so $\max\langle s_A\rangle = n + 2|s_A| - 2$

suppose s' has one less A smoothing than s

then $\alpha(s') - \beta(s') = \alpha(s) - \beta(s) - 2$

and $|s'| = |s| \pm 1$ (depending if circles merged or split)

$$\text{so } \max\langle s'\rangle = \max\langle s\rangle - 2 \pm 2 = \begin{cases} \max\langle s\rangle \\ \text{or} \\ \max\langle s\rangle - 4 \end{cases}$$

given any state s you get it from s_A by switching some A smoothings to B smoothings

we can do this one-by-one to get states

$s_0 = s_A, s_1, \ldots s_k = s$

where each $s_i \to s_{i+1}$ swithes one A to a B

from above we see $\max\langle s_A\rangle \geq \max\langle s_1\rangle \geq \ldots \geq \max\langle s_k\rangle = \max\langle s\rangle$

so all terms in $\langle D\rangle$ have degree $\leq \max\langle s_A\rangle = n + 2|s_A| -$

note: since $\max\langle s_A\rangle = n + 2|s_A| - 2$ then $\max\langle D\rangle$ will also be $n + 2|s_A| - 2$ unless the term in $\langle s_A\rangle$ is cancelled by a term of same degree in $\langle s\rangle$ for some s

if D A-adequate then for any state s differing from s_A by changing one A to a B we have

$|s| < |s_A|$

so from above $\max\langle s\rangle < \max\langle s_A\rangle$

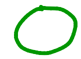
$\therefore$ from above any state different from s_A

satisfies this so the $n+2|s_A|-2$ term in $\langle s_A\rangle$

can't be cancelled in $\langle D\rangle$ and $\max\langle D\rangle = n+2|s_A|-2$

The statement for $\min\langle D\rangle$ and B-adequate diagrams is similar (or same if you consider $m(D)$)

II) We prove $|s_A|+|s_B| \leq n+2$ by induction on n

Base case: $n=0$ then we have  D

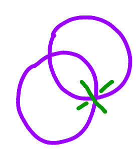
only one state so $s_A = s_B = D$

and $|s_A|+|s_B| = 2 = 0+2$

Inductively assume true for diagram with $n-1$ crossings

let D be a diagram with n crossings

fix one crossing c and notice that for at least one choice of smoothing resulting diagram D' is connected

think about this if you don't see it right away 

suppose this was an A-smoothing (other case similar)

then s_A for D and $\underbrace{s_A \text{ for } D'}_{\text{we call this } s_A' \text{ for clarity}}$

are the same: $s_A = s_A'$

as discussed above $|s_B| = |s_B'| \pm 1$

inductive hypothesis

thus $|s_A|+|s_B| = |s_A'|+|s_B'| \pm 1 \leq (n-1)+2 \pm 1 \leq n+2$

so done

we are left to see $|s_A|+|s_B| = n+2$ if D alternating

we delay this untill later (need Euler characteristic)

F. Lecture Supplement: Other polynomials

so what is the most general polynomial invariant you can define using the skein relation above?

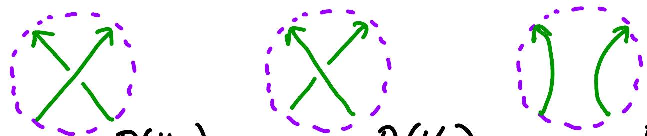
it is not too hard to show there is a unique function

$$\{\text{oriented links}\} \longrightarrow \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$$
$$K \longmapsto P_L$$

$\leftarrow$ Laurent polys in x, y, z

satisfying A) $P_{\text{unknot}} = 1$

B) if K_+, K_-, and K_0 have diagrams related by



$D(K_+)$ $D(K_-)$ $D(K_0)$

then $$x P_{K_+} + y P_{K_-} + z P_{K_0} = 0$$

Substituting x, y, z for any polynomials in any variables gives a polynomial satisfying a skein relation, and any such polynomia comes from this. So P_K is "the most general" skein polynomia

e.g. $\Delta_K(t) = P_K(1, -1, t^{-1/2} - t^{1/2})$

$V_K(t) = P_K(t^{-1}, -t, t^{-1/2} - t^{1/2})$

exercise:

Show P_K is a homogeneous polynomial

Given this we can turn P_K into a non-homogeneous polynomial in 2 variables (just set one of the variables equal to a function of the others)

the most common way this is done is by setting

$x = \alpha^{-1}$, $y = -\alpha$, $z = -z$

or

$x = \ell$, $y = \ell^{-1}$, $z = m$

Any of these 2 variable polynomials is called the HOMFLY (or HOMFLY-PT or FLYPMOTH or the Generalized Jones) polynomial

it was discovered in the late 1980s by 2 groups

Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter

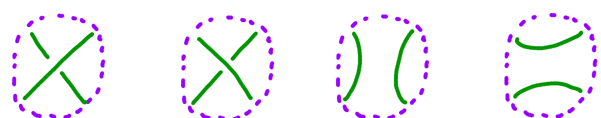
Przytycki and Traczyk

There is also a generalization of the Jones polynomial as follows

Fact: There exist a unique function

$$[\cdot]: \left\{ \begin{array}{c} \text{unoriented} \\ \text{link diagrams} \end{array} \right\} \longrightarrow \mathbb{Z}[z^{\pm 1}, a^{\pm 1}]$$

such that (1) if D_+, D_-, D_0, D_∞ are related by



D_+ D_- D_0 D_∞

then $[D_+] + [D_-] = z([D_0] + [D_\infty])$

(2)

(3) $[\text{unknot}] = 1$

(4) if D and D' are related by Reidemeister type 2 or 3 moves then $[D] = [D']$

now just as we did to get a link invariant out of the Kauffman bracket we define

$$K_L(a,z) = a^{-\omega(D)}[D]$$ where D is a diagram for L and recall ω is the writhe

K_L is called the Kauffman polynomial of L

(note this is an invariant of oriented links! because of $\omega(D)$)

recall the Kauffman bracket satisfies

$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$$

so $\langle \times \rangle + \langle \times \rangle = (A+A^{-1})(\langle)(\rangle + \langle \asymp \rangle)$

and $\langle \gamma \rangle = -A^{3}\langle) \rangle$ and $\langle \gamma \rangle = -A^{3}\langle) \rangle$

$\therefore \quad F_L(A) = K_L(A+A^{-1}, -A^{3})$

and hence $\boxed{V_L(t) = K_L(t^{1/4}+t^{-1/4}, -t^{3/4})}$

exercises:

1) $K_{\bar{L}} = K_L$ if $\bar{L}$ is L with orientation reversed

2) $K_{m(L)}(z,a) = K_L(z,a^{-1})$

3) $K_{O_k} = ((a+a^{-1})z^{-1}-1)^{k-1}$ where O_k: k component unl.: $\underbrace{O\ O \cdots O}_{k}$

4) $K_{L_1 \cup L_2} = ((a+a^{-1})z^{-1}-1)\, K_{L_1}\cdot K_{L_2}$

where $L_1 \cup L_2$ is just union L_1 and L_2 where they are separated by an $\mathbb{R}^2$

5) $K_{L_1 \# L_2} = K_{L_1}\cdot K_{L_2}$ where

$L_1 \# L_2$ called a connect sum of L_1 and L_2

We have now seen all the "mainstream" polynomials (there is also the A-polynomial, but quite different)

let's end with seeing how to prove Th$^{\underline{m}}$ 5. II) using K

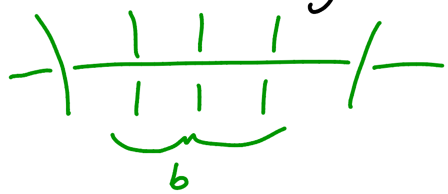
recall Th$^{\underline{m}}$ 5. II) says

> If D and D' are reduced alternating diagrams for an oriented link L then $\omega(D) = \omega(D')$

K_L is a polynomial in z and a

we can write it as $[D] = \sum p_i(a) z^i$ where $p_i(a) \in \mathbb{Z}[a^{\pm 1}]$

given a diagram D we say its bridge length, $b(D)$, is the maximu[m] number of consecutive overcrossings in D



note, if # crossings of $D > 0$, then $b(D) > 0$

lemma 9:

> let D be a link diagram with n crossings and bridge length b.
> Then
> $$\text{max degree in } z \text{ of } K_D \leq n-b$$

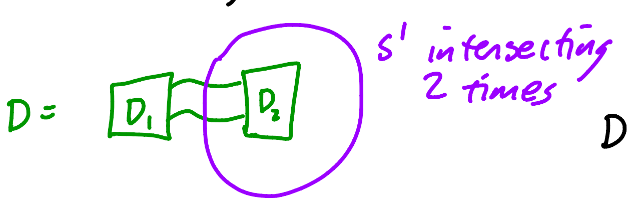
note:

1) lemma is equivalent to $p_i(a) = 0, \ \forall \ i > n-b$
2) if D is alternating, then $b(D) = 1$ so $n(D) - b(D) = n-$

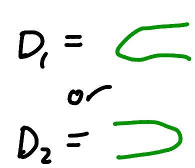
lemma 10:

> let D be a connected alternating diagram with $n \geq 2$ crossings
> Then (1) $p_{n-1}(a) = r(a - a^{-1}) \quad r \in \mathbb{Z}, r \geq 0$
> (2) if D is prime and reduced, then $r > 0$

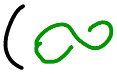
here D prime means if $\exists$ a circle in $\mathbb{R}^2$ intersecting D in 2 places (and transversely), then on one side of the circle D is just an arc (i.e. no crossings)



D prime $\Rightarrow$



(i.e. prime means not a non-trivial connect sum)

given any D we can write it as $D = D_1 \# \ldots \# D_\ell$ where D_i are pri[me] exercise: if D is connected and prime, then D reduced or has only one crossing ( ,  , ...)

Proof of Th^{m} 5.II):

let D and D' be connected, reduced, alternating diagrams for same link

$$a^{\omega(D)}[D] = K_D = K_{D'} = a^{\omega(D')}[D']$$

so $[D'] = a^{\omega(D)-\omega(D')}[D']$

we can write $D = D_1 \# \dots D_\ell$ with D_i prime
and $D' = D_1' \# \dots D_{\ell'}'$ with D_i' prime

compair the coefficient of the higest power of z:

$$\prod r_i'(a+a^{-1})^{m_i'} = a^{\omega(D)-\omega(D')} \prod r_i (a+a^{-1})^{m_i}$$

the r_i and $r_i' \neq 0$ by lemma 7

(here we use that higest power of K_{D_i} is $r_i(a+a^{-1})$ with $r_i \neq 0$ from lemma 7 and exercise above: $K_{L_1 \# L_2} = K_{L_1} \cdot K_{L_2}$)

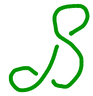
note left hand side symmetric in a, a^{-1} but right hand side wont be if $\omega(D) \neq \omega(D')$

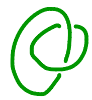
$\therefore \omega(D) = \omega(D')$ #

Proof of lemma 10 given lemma 9:

we induct on n

Base case $n=2$: 2 possibilities (upto mirroring)

if $K =$  $F_K = 1$ $\quad p_{2-1}(a) = p_1(a) = 0(a+a^{-1})$ ✓

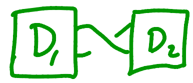
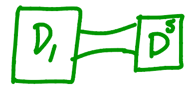
if $K =$  then check $[K] = (a+a^{-1})z + 1 - (a+a^{-1})z^{-1}$

so $p_{2-1}(a) = p_1(a) = 1(a+a^{-1})$ ✓

so (1) and (2) true for $n=2$ ✓

inductive step for (1):

if D not reduced let D' be diagram with one of the reducing crossings removed

eg. $D =$  $\Rightarrow D' =$ 

now D, D' diagrams for same link so

$$a^{\omega(D)}[D] = F_D = F_{D'} = a^{\omega(D')}[D']$$

you can check $\omega(D) = \omega(D') \pm 1$ so

$$[D] = a^{\pm 1}[D']$$

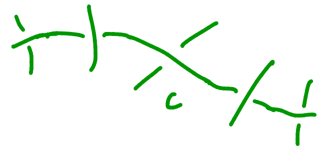
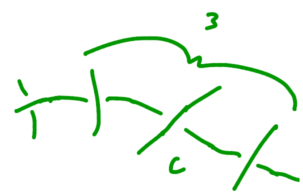
but D' has $n-1$ crossings so $p_{n-1}(a) = 0$ ✓

if D is reduced then let $D_+ = D$ and D_-, D_0, D_∞ the corresponding diagrams

focus on any crossing c

D reduced $\Rightarrow D_0, D_\infty$ are connected diagrams

note: D_- has bridge length 3

D_+  $\rightarrow D_-$ 

so by lemma 6, $p^-_{n-1}(a) = 0$ (here $p^-_i(a)$ is from coeff of $[D_-]$ and similarly p^0_i and p^∞_i

we have $[D] = z([D_0] + [D_\infty]) - [D_-]$

$$\therefore \; p_{n-1}(a) = p^0_{n-2}(a) + p^\infty_{n-2}(a)$$

$$= (r^0 + r^\infty)(a + a^{-1}) \qquad \text{and } r^0 + r^\infty \geq 0$$

induction since D^0, D^∞ have $n-1$ crossings and are alternating

inductive step for (2):

exercise: Show D prime $\Rightarrow D_0$ or D_∞ prime (and therefore reduced by exercise above)

$\therefore$ from above $p_{n-1}(a) = (r^0 + r^\infty)(a + a^{-1})$ and by induction one of r^0 or $r^\infty > 0$ ▦

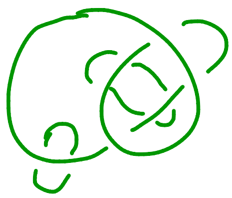
Proof of lemma 4:

proof is by induction on $(n, n-b)$ (here $(a,b) < (c,d)$ $\Leftrightarrow a < c$ or ($a = c$ and $b < d$))

base case $n-b=0$:

exercise: In this case D is a diagram for O_k

Hint: show you can traverse one component of D s.t. the first time you hit each crossing its an undercrossing and all other components trivial unknots



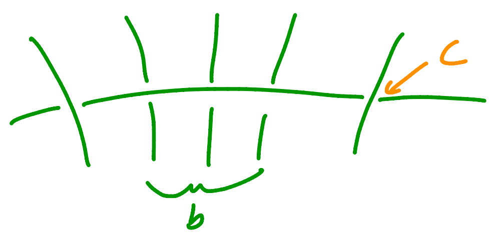
$$\therefore [D] = a^{\omega(D)} K_{O_k} = a^{\omega(D)} \left((a+a^{-1}) z^{-1} - 1 \right)^{k-1}$$

so no positive powers of z

i.e. $\rho_i(a) = 0 \quad \forall\, i > 0 = n-b.$

inductive step $n-b>0$:

consider a crossing c at one end of a bridge of D of length b so that bridge does not pass through c



let $D_+ = D$ (focus on c) and D_-, D_0, D_∞ the related diagrams

note: D_- has bridge length $\geq b+1$ and D_0, D_∞ have $\leq n-1$ crossings and bridge length $\geq b$

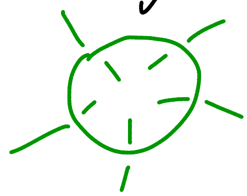
by induction $p_i^- = p_i^0 = p_i^\infty = 0$ if $i > n-b-1$

now $[D] = z([D_0] + [D_\infty]) - [D_-]$

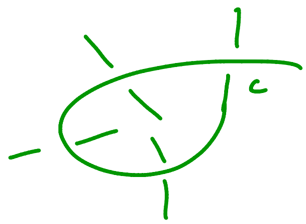
$\therefore p_i = 0$ if $i \geq b-n$

so we are done unless there is no such c

this happens when the brige is one component of D



or when bridge contains c again



in the first case D is equivilent to D' (via Reidemeister II) and III) moves) where D' has less crossings

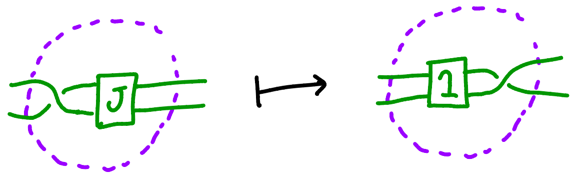
$\therefore$ $[D]=[D']$ and done by induction

In the second case can get D' by elliminating loop and $[D]=a^{\pm 1}[D']$ and D' has $<n$ crossings

so done by induction #

Remark: For the record Tait had 3 main conjectures. The 2 above in Th$^{\underline{m}}$ 5 and the following one:

a flype is the following operation on link diagrams



or with crossings switched

Tait conjecture III:

> If D_1 and D_2 reduced alternating diagrams of oriented L
> Then D_1 and D_2 are related by flypes

note: 1) Tait III $\Rightarrow$ Tait II

2) Tait III proved by Menasco-Thistlethwaite 1991